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COMMENT

Particle trapping near a parallel rotator

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Abstract. The Störmer theory of charged particles in a magnetic dipole field has been generalised by superposing an axially symmetric electric field. A formula describing the borders between 'allowed' and 'forbidden' regions for a particle with a given azimuthal component of generalised momentum is derived. The theory has been tested by some numerically calculated proton orbits in the fields of a parallel rotator.

The study of so-called 'allowed' and 'forbidden' regions of the Störmer problem (Störmer 1955) has recently been generalised by superimposing an electric potential with azimuthal symmetry on the vector potential (Schuster and Thielheim 1987) of a magnetised sphere.

The vector potential $\boldsymbol{A}$ of a magnetic dipole $\boldsymbol{\mu}$ with an additional scalar potential A_0 may be expressed with the help of cylindrical coordinates (R, φ, z),

$$\boldsymbol{A} = \begin{pmatrix} A_R \\ A_\varphi \\ A_z \end{pmatrix} = \begin{pmatrix} 0 \\ \mu \dfrac{R}{(R^2+z^2)^{3/2}} \\ 0 \end{pmatrix} \qquad A_0 = A_0(R, z) \tag{1}$$

assuming that the z axis and the magnetic vector $\boldsymbol{\mu}$ are parallel and that the dipole is at rest in a given inertial frame of reference. In what follows v is the velocity of a particle with rest mass m and electric charge e; also $\gamma = (1 - v^2/c^2)^{-1/2}$ is the corresponding Lorentz factor, and a dot denotes derivation with respect to the time coordinate t. Since the Lagrangian

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}} + \frac{e\mu}{c}\frac{R^2}{(R^2+z^2)^{3/2}}\dot{\varphi} - eA_0 \tag{2}$$

is independent of the time coordinate t and the azimuthal coordinate φ, there exist two constants of motion, namely the energy

$$E = \gamma mc^2 + eA_0 \tag{3}$$

and the azimuthal component of the generalised momentum

$$p_\varphi = m\gamma R^2\dot{\varphi} + \frac{e\mu}{c}\frac{R^2}{(R^2+z^2)^{3/2}}. \tag{4}$$

With the definitions $v_\varphi = R\dot{\varphi}$, $r = \sqrt{R^2+z^2}$, $R = r\cos\vartheta$ and $E_R = mc^2$, equations (3) and (4) lead to expressions for the total velocity and the corresponding azimuthal component of particle velocity (assuming $\cos\vartheta \neq 0$), whereas the condition $v_\varphi^2 \leqslant v^2$ delivers the inequality

$$r^4[E^2 - E_R^2 + (eA_0)^2 - 2eA_0E]\cos^2\vartheta - (cp_\varphi)^2r^2 + 2cp_\varphi e\mu r\cos^2\vartheta - (e\mu)^2\cos^4\vartheta \geqslant 0 \tag{5}$$

which provides for a restriction of particle motion. If the electric potential of a quadrupole field (with quadrupole moment Q)

$$A_0 = \frac{Q}{4r^3}(3\sin^2\vartheta - 1) \tag{6}$$

is inserted into relation (5), the latter takes the form†:

$$f(r, \cos^2\vartheta) := r^6(E^2 - E_R^2)\cos^2\vartheta - r^4(cp_\varphi)^2 + r^3[2e\mu cp_\varphi - \tfrac{1}{2}eQE(3\sin^2\vartheta - 1)]\cos^2\vartheta$$
$$- r^2(e\mu)^2\cos^4\vartheta + \frac{(eQ)^2}{16}(3\sin^2\vartheta - 1)^2\cos^2\vartheta \geqslant 0. \tag{7}$$

A second restriction imposed on particle coordinates (r, ϑ) for given parameters E, E_R and Q originates from the fact that kinetic energy plus rest energy must be positive:

$$\frac{E - eA_0}{E_R} = \gamma \geqslant 1 \tag{8}$$

corresponding to

$$(E - E_R)r^3 - \frac{eQ}{4}(3\sin^2\vartheta - 1) \geqslant 0. \tag{9}$$

The equation

$$f(r, \cos^2\vartheta) = 0 \tag{10}$$

is a polynomial of sixth order in r, but only of third order in $\cos^2\vartheta$. So the borders between 'forbidden' and 'allowed' regions, defined by (10), can be represented with the help of Cardano's formula.

For $Q \neq 0$ the direct vicinity $(r \to 0)$ of the dipole is always 'allowed', and so is the distant region $(r \to \infty)$. Regarding (10) as a function of r, there may be found ranges of values of the parameters p_φ, μ, Q, $\cos^2\vartheta$ for which there exist four real, positive roots (with the help of Descartes' theorem; see e.g. Obreschkoff (1963)). This is an indication for the existence of an 'allowed' range of r enclosed by two 'forbidden' ranges, which themselves are enclosed by two allowed ranges. In particular, there exist real trapping tori for certain parameter values. In the latter case equation (9), which must also hold for all physical configurations does not lead to a further restriction of these allowed regimes. The electric quadrupole moment can be thought of to be created by the rotation of a homogeneously magnetised, ideally conducting sphere, with its axis of rotation parallel to its magnetic dipole moment (Deutsch 1955). This interpretation is of special interest here, since it is often used as a simplified model for a rotating neutron star. The trapping region predicted here has been tested numerically by calculating proton orbits in the field of the parallel rotator with appropriately chosen parameters p_φ, μ, Q (figure 1).

† Unfortunately in the aforementioned paper (Schuster and Thielheim 1987) the sign of the last term in the relation corresponding to (7) is not correct, which affects the subsequent conclusions and some of the diagrams.

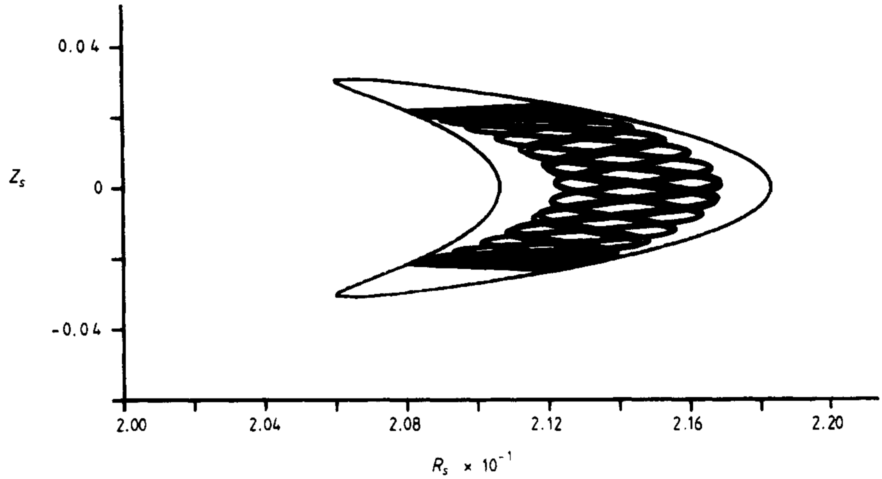

Figure 1. Example of a proton orbit in a trapping torus. $\varepsilon = 2.0$, $\lambda = 0.2$, $\kappa = -0.00235$. The dimensionless parameters are chosen according to the definitions in Schuster and Thielheim (1987).

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